PROJECTED WRITTEN NOTES FROM THE M408D LECTURE ON THURSDAY, APRIL 4, 2024, ON Sec 14.3: Interpretations of the PARTIAL DERIVATIVE VALUES, SEC 14.5: THE MULTI-VARIABLE CHAIN RULE, and Sec 14.4: The Differential dz

CLASS # 22

Recall the Partial Derivatives

Problem: For $f(x,y) = x^3 + x^2y^3 - 2y^2$,

find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and

also $f_x(2,1)$ and $f_y(2,1)$.

Sol'n: $\frac{\partial f}{\partial x} = 3x^2 + 2xy^3$

$\frac{\partial f}{\partial y} = 3x^2y^2 - 4y$

$f_x(2,1) = 3\cdot 2^2 + 2\times 2\times 1^3 = 16$

$f_y(2,1) = 3\times 2^2 \times 1^2 - 4\times 1 = 8$

What do these values mean?

Interpretation of Partial Derivative Values:

$f_x(x_0, y_0)$ and $f_y(x_0, y_0)$

Two Types:

1. As a slope of a tangent line.
2. As a Rate of change.

$f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ as a Slope of a Tangent LINE

PARTIAL DERIVATIVES AS SLOPES OF TANGENT LINES

THE BLUE SURFACE is the GRAPH of $z = f(x,y)$.

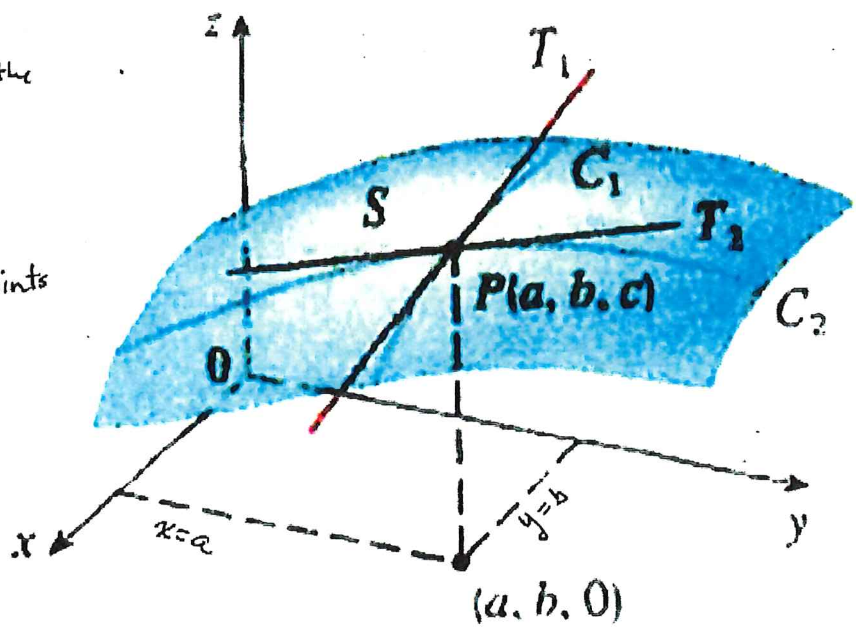
$(a,b,0)$ is a point in the xy plane $(z=0)$.

Point $P(a,b,c)$ has $c = f(a,b)$.

Curve C_1 consists of the points on the intersection of ...e surface graph ...h the vertical plane $y=b$

T_1 is the tangent line, tangent to curve C_1 at the point $P(a,b,c)$.

The slope of the tangent line T_1 is $f_x(a,b) = \left.\dfrac{\partial z}{\partial x}\right|_{\substack{x=a\\ y=b}}$.



C_2 consists of the points on the intersection of the surface graph with the vertical plane $x=a$.

T_2 is the tangent line, tangent to curve C_2 at the point $P(a,b,c)$.

The slope of the tangent line T_2 is $f_y(a,b) = \left.\dfrac{\partial z}{\partial y}\right|_{\substack{x=a\\ y=b}}$.

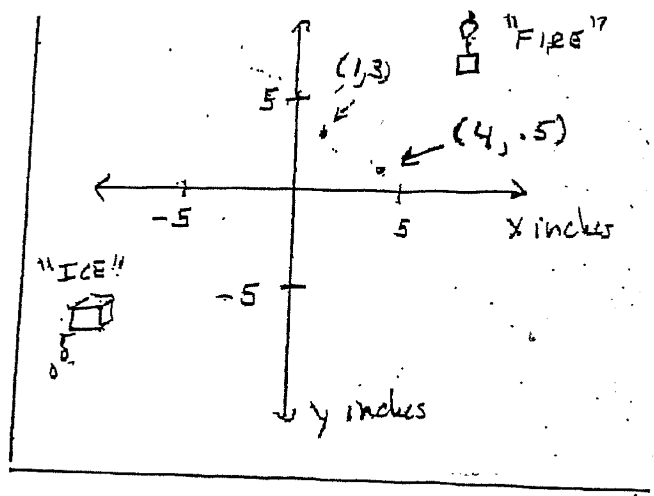
FIGURE 1
The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.

p. 964

And As a Rate of Change of $f(x, y)$

The "Rate of Change" Interpretation of the Partial Derivative

A Temperature Function and Its Partial Derivatives



Sources of heat are placed in QUADRANT I and sources of COLD ARE PLACED in QUADRANT III in such a manner that,

at every point (x,y) with $|x| < 5$ inches and $|y| < 5$ inches,

the Temperature $T(x,y)$ at that point is given by the function $T(x,y) = 70 + 3x + x^3 + 2y^3$ Degrees Farenheit ($^\circ$)

The partial derivatives of T are:

$$\frac{\partial T}{\partial x} = T_x(x,y) = 3 + 3x^2 \text{ Degrees per inch}$$

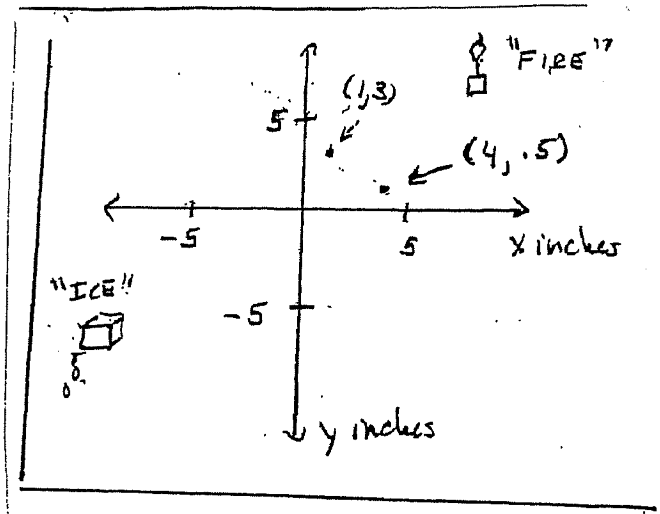
$$\frac{\partial T}{\partial y} = T_y(x,y) = 6y^2 \text{ Degrees per inch}$$

RECALL:

$T(x,y) = 70 + 3x + x^3 + 2y^3$

$T_x(x,y) = 3 + 3x^2$

$T_y(x,y) = 6y^2$



At the point $(x,y) = (1,3)$,

$T_x(1,3) = 6°$ per inch

$T_y(1,3) = 54°$ per inch

x	y	$T(x,y)$
1.00	3	128.000000
1.01	3	128.060301

$\Delta x = 0.01$ and $\Delta T = 0.060301$, and $T_x(1,3) = 6$

$T_x(1,3) \times \Delta x = 6 \times (.01) = 0.06 \approx 0.060301 = \Delta T$

$T_x(1,3) \cdot \Delta x \approx \Delta T$

x	y	$T(x,y)$
1	3.00	128.000000
1	3.01	128.541802

$\Delta y = 0.01$ and $\Delta T = 0.541802$, and $T_y(1,3) = 54$.

$T_y(1,3) \times \Delta y = 54 \times (.01) = 0.54 \approx 0.541802 = \Delta T$

$T_y(1,3) \cdot \Delta y \approx \Delta T$

Second Order Partial Derivatives of $z = f(x,y)$

$\frac{\partial z}{\partial x} = f_x(x,y)$ | $\frac{\partial z}{\partial y} = f_y(x,y)$

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{x,x}(x,y)$ | $\frac{\partial^2 z}{\partial y^2} = f_{yy}(x,y)$

$\frac{\partial^2 z}{\partial y\,\partial x} = f_{xy}(x,y)$ $\qquad$ $\frac{\partial^2 z}{\partial x\,\partial y} = f_{yx}(x,y)$

If all first partial are continuous
then the two mixed partials are equal

For $z = f(x,y) = x^3 y^4$

$\frac{\partial z}{\partial x} = 3x^2y^4$ | $\frac{\partial z}{\partial y} = 4x^3y^3$

$\frac{\partial^2 z}{\partial x^2} = 6xy^4$ | $\frac{\partial^2 y}{\partial y^2} = 12x^3y^2$

$f_{xy}(x,y) = 12x^2y^3$ | $f_{yx}(x,y) = 12x^2y^3$

The Multi-Variable CHAIN RULE

TO REVIEW:

The Chain Rule for functions of one variable

Let $y = 5\cos(t^3+1) = f(t)$

$f(t) = y = 5\cos(u)$, where $u = t^3 + 1$.

$f'(t) = \frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} = -5\sin(u)(3t^2)$

$$\boxed{\frac{dy}{dt} = -15t^2\sin(t^3+1)}$$

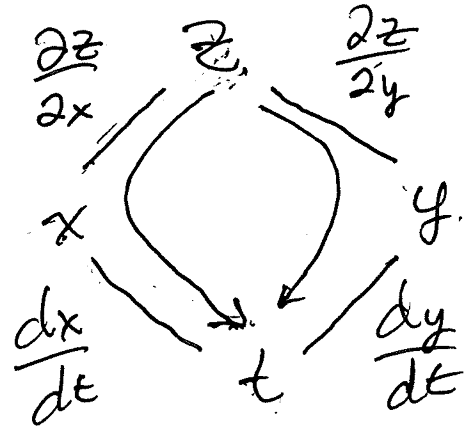
The Multivariable CHAIN Rule (Case 1)

Given $z = f(x,y)$ and $x = g(t)$ and $y = h(t)$, $z = f(x,y) = f(g(t), h(t))$ is a function of t, with its derivative $\frac{dz}{dt}$, and, in this case,

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}}$$

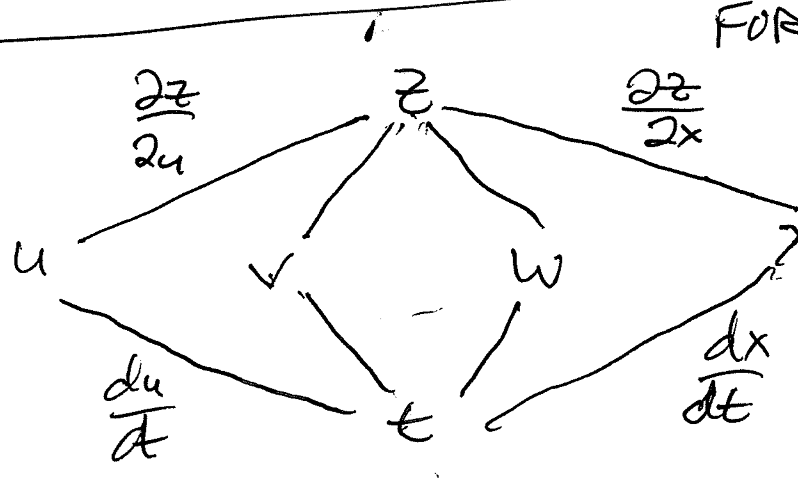
Memory Aid:

Intermediate variables →



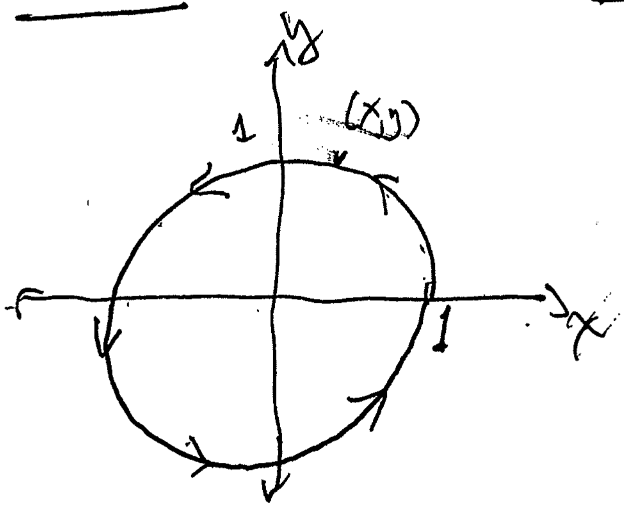
$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

FOR 3 or More Intermediate Variables



$$\frac{dz}{dt} = \frac{\partial z}{\partial u} \cdot \frac{du}{dt} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dt} + \frac{\partial z}{\partial w} \cdot \frac{dw}{dt} + \frac{\partial z}{\partial x} \cdot \frac{dx}{dt}$$

Problem:



The position (x,y) of the particle is given by $x = \cos(2\pi t)$, $y = \sin(2\pi t)$ at time t seconds.

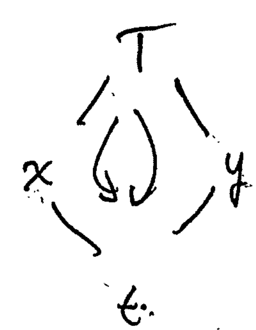
The Temperature T at the point (x,y) is $T(x,y) = (70 + 3x + 4y)\,^\circ F$.

View $T = \text{Temp} = f(t) =$ The Temperature of the particle at time t sec.

Find $\dfrac{dT}{dt} = f'(t)$

Soln.

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial T}{\partial y}\cdot\frac{dy}{dt}.$$



$$\frac{dT}{dt} = (3)\left(-2\pi\sin(2\pi t)\right) + (4)\left(2\pi\cos(2\pi t)\right)$$

$$\boxed{\frac{dT}{dt} = -6\pi\sin(2\pi t) + 8\pi\cos(2\pi t)}$$

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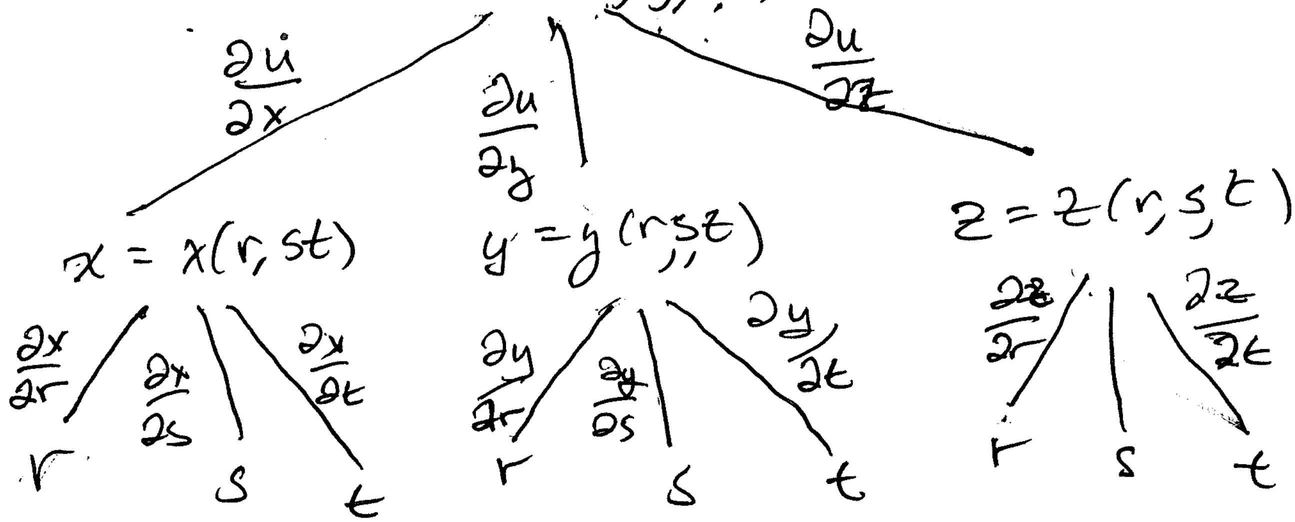
The Multi-variable Chain Rule (General Case)

Given $u = f(x,y,z)$



$u = H(r,s,t)$

$$H_r = \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial r}.$$

$$H_s = \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial s}$$

$$H_t = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \ldots$$

Ex: Let $u = x^4 y + y^2 z^3$ where

$x = rse^t, \quad y = rs^2 e^{-t}, \quad z = r^2 s (\sin t)$.

Find the value of $\frac{\partial u}{\partial s}$ when $r=2, s=1, t=0$

Sol'n
$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$\frac{\partial u}{\partial s} = (4x^3 y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2 \sin t)$$

When $r=2, s=1, t=0,$
$x=2, \quad y=2, \quad z=0$

$$\frac{\partial u}{\partial s} = (64)(2) + (16)(4) + (0)$$
$$128 + 64 = 192$$

$$\left.\frac{\partial u}{\partial s}\right|_{\substack{r=2\\ s=1\\ t=0}} = 192$$

The Total Differential dz

Let $z=f(x,y)$ be a function of two variables.

Given Initial Input (x_0,y_0)

make a change to the input.

which gives an initial output $z_0=f(x_0,y_0)$.

The FINAL Input is (x_1,y_1)

and it gives a final output $z_1=f(x_1,y_1)$

Then, write $\Delta x = x_1 - x_0$, $\Delta y = y_1 - y_0$

and $\Delta z = z_1 - z_0$

The (TOTAL) Differential dz is

the function (For fixed values of x and y)

of two variables dx and dy,

given by:

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

or

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy.$$

Ex: For $z = f(x,y) = 5x^3y + xy^2$,

$f_x(x,y) = (15x^2y + y^2)$ and $f_y(x,y) = (5x^3 + 2xy)$

$$\boxed{dz = (15x^2y + y^2)\,dx + (5x^3 + 2xy)\,dy}$$

The Total Differential Principle

In any application of dz,

Set $(x,y) = (x_0, y_0)$ = Initial Inputs

and $dx = \Delta x = x_1 - x_0$

$dy = \Delta y = y_1 - y_0$

With $\Delta z = z_1 - z_0 = f(x_1,y_1) - f(x_0,y_0)$

$\underline{dz \simeq \Delta z}$.

The value of the differential dz is a good approximation of Δz = the change in function value.

Problem: Let $z = 5x^3y + xy^2$

Use the differential dz to Approximate Δz that results as the input changes from $(x,y) = (1,2)$ to $(x,y) = (1.3, 1.9)$

Sol'n:
$$\Delta x = 1.3 - 1.0 = 0.3$$
$$\Delta y = 1.9 - 2.0 = -0.1$$

$$dz = (15x^2y + y^2)\,dx + (5x^3 + 2xy)\,dy$$

Set $(x,y) = (1,2)$ = Initial Inputs

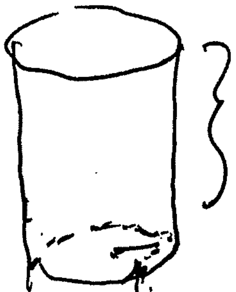
Set $dx = \Delta x = 0.3$, $dy = \Delta y = -0.1$

$$dz = (34)(0.3) + (9)(-0.1) = 9.3 =$$
$$dz = 9.3$$

By the Principle, $dz \doteq \Delta z$

$$\Delta z = 11.5645$$
$$dz = 9.300$$

Problem: A right circular cylinder starts with a radius $r_0 = 5$ cm and height $h_0 = 15$ cm. Initially.



$\} \, h_0 = 15$ cm

$r_0 = 5$ cm

It grows to where
$r_1 = 5.05$ cm
and
$h_1 = 16.5$ cm.

ΔV = the change in Volume V that results.

Task: Use the differential dV to approximate ΔV.

Sol'n: $V = \pi r^2 \cdot h = f(r, h)$

$\left.\begin{array}{l} r_0 = 5.00 \text{ cm} \\ r_1 = 5.05 \text{ cm} \end{array}\right\} \Delta r = 0.05 \text{ cm}$

$\left.\begin{array}{l} h_0 = 15.0 \text{ cm} \\ h_1 = 16.5 \text{ cm} \end{array}\right\} \Delta h = 1.5 \text{ cm}$

$$\boxed{dV = (2\pi r h) \cdot dr + (\pi r^2) dh}$$

Set $(r, h) = (5, 15)$ $\quad r = 5, h = 15$ (Initial Inputs)

$dr = \Delta r = 0.05$ cm, $\quad dh = \Delta h = 1.5$ cm

$$dV = \{(2\pi)(5)(15)(0.05) + (\pi)(5^2)(1.5)\} \text{ cm}^3$$

$$dV = 141.37 \text{ cm}^3$$

$$\Delta V = 143.90 \text{ cm}^3$$